where

$$\beta = [\alpha(T_{s,\text{vac}} - T_s)\tan a/a]/[\alpha(T_s - T_r) + 1]$$

Calibration and Discussion

Figure 1 shows the variation of $\bar{R}$ as a function of $1/s$ for various values of the parameter a for $(T_{s,\text{vac}} - T_s) = 0$. It can be seen that for a fairly wide range of values of a, the parameter $\bar{R}$ is independent of a. Furthermore, it can be shown that for small values of a, $\bar{R}$ approaches the limit 1.2 as $1/s$ increases.

The effect of overheat, $\tau = (T_w - T_a)/T_a$, on Nusselt Number for $\tau < 0.4$ is found to be small,[1,4] and therefore $1/s = \xi$. For most experiments in low density, unheated flows, the Nusselt Number correction factor $\psi_N > 0.6$ and $\xi = 1/s > 0.4$. From Fig. 1 for $1/s > 0.4$, $\bar{R}$ is virtually a constant. In practice, this value of $\bar{R}$ is obtained from a freestream calibration. Hence the measured values of adiabatic temperature and Nusselt Number can be corrected directly to give the values for an infinitely long wire, without having to measure the diameter and effective length of the wire, and without having to use an iteration process.

The freestream calibration was performed in a Mach 6, unheated low-density airstream.[3] The hot wires were platinum-plated tungsten of nominal diameter 0.0002 in. and the distance between the support needles was approximately $\frac{3}{8}$ inch. Thermocouples were soldered to within 0.005 in. of the tip of one of the support needles, so as to measure T_s directly. For any given wire, Nusselt Number is proportional to $(1/k_0)(dR/dI^2R)^{-1}$. Furthermore, for free molecule flow, T_a and Nu_0 are given by the following relationships[7]:

$$Nu_0 = [(\gamma - 1)/2\pi^{3/2}]\bar{\alpha}Pr_0Re_{0,d}g(S)/S \qquad (15)$$

and

$$T_a/T_0 = \{1 + [(\gamma - 1)/2]M^2\}^{-1}f(S)/g(S) = F(S) \qquad (16)$$

where $f(S)$ and $g(S)$ depend only on the speed ratio S and the number of excited degrees of freedom. These functions are tabulated in Ref. 7 for both monatomic and diatomic perfect gases. The functions $g(S)/S$ and $F(S)$ approach constants for $S > 2.5$. The limiting values are

$$\left[\frac{g(S)}{S}\right]_{S\to\infty} = \frac{\gamma + 1}{\gamma - 1}\pi^{3/2} \text{ and } [F(S)]_{S\to\infty} = \frac{2\gamma}{\gamma + 1}$$

Experimental investigations of the Nusselt Number and adiabatic temperature by Stalder et al.,[7,8] Dewey,[1] Vrebalovich,[4] and Atassi and Brun[5] indicate that there is excellent agreement with the free molecule theory.

For any given wire, Eq. (15) can be written as

$$(1/k_0)(dR/dI^2R)^{-1}_{\text{corr}} = CPr_0(\rho u/\mu_0)g(S)/S$$

where the constant C is a function of the accommodation coefficient $\bar{\alpha}$, wire length l and wire diameter d. This constant C can be obtained for any particular wire from a freestream calibration of $(1/k_0)(dR/dI^2R)^{-1}_{\text{corr}}$ against $\rho u/\mu_0$, since the function $g(S)/S$ is a constant for $S > 2.5$. In the calibration process, since $M \simeq 6$, Ta/T_0 is taken to approach its asymptotic limit of $2\gamma/(\gamma + 1)$. Hence from Eqs. (5) and (7), ξ tanhl/ξ and ψ_N can be calculated. The calibration yields two quantities; 1) the slope of $(dR/dI^2R)^{-1}_{\text{corr}}$ against ρu or the slope of $(1/k_0)(dR/dI^2R)^{-1}_{\text{corr}}$ against $\rho u/\mu_0$ and 2) the value of $\bar{R}$ for the particular wire. This calibrated value of $\bar{R}$ is expected to take into account the small variations of $(T_{s,\text{vac}} - T_s)$. In the study of wakes behind circular cylinders[3] it was found that $T_s/T_{0,fs}$ is virtually a constant in the wake (though T_s/T_0 exhibits large gradients). Also $T_{s,\text{vac}} = T_{0,fs}$ (unheated flow) and therefore the value of $(T_{s,\text{vac}} - T_s)$ will be approximately the same for freestream and the wake and is of the order 5°C.

Figure 2 shows the calibration of $(dR/dI^2R)^{-1}_{\text{corr}}$ against ρu for two different hot wires. The least squares straight lines through the data points approximately pass through the origin, which indicates excellent agreement with free molecule theory. In Fig. 3 the measured $\bar{R}$ is plotted against ξ and compared with the theoretical prediction for $(T_{s,\text{vac}} - T_s) = 0$. The parameter a is of the order 10^{-1} for most probes. This correction procedure has been used to obtain flow parameters in the near wake of a circular cylinder.[3]

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Optimal Design of Statically Determinate Beams Subject to Displacement and Stress Constraints

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Introduction

As was pointed out by Barnett[1] and stressed by Huang and Tang[2] the optimal design of a beam that is only subject to deflection constraint may not be a well-posed problem. Consider for instance the beam shown in Fig. 1, that carries a concentrated load P and a couple C and which has to be designed for minimum weight, the behavioral constraint being an upper bound on the displacement at the center of the span. Because load and couple have opposite effects on this displacement, it is possible to find a design with vanishing displacement at this point. This property will be conserved when all bending stiffnesses are scaled down by an arbitrary factor; this means that the behavioral constraint may be fulfilled by a design of arbitrarily small structural weight. Of course, this solution of the optimization problem is physically meaningless because the stresses at some cross sections become arbitrarily great. To preclude this possibility, we have

Received November 30, 1970; revision received February 1, 1971. Research sponsored by the Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio, under Contract F33615-69-C-1826. The author is indebted to W. Prager, Brown University, for valuable suggestions.

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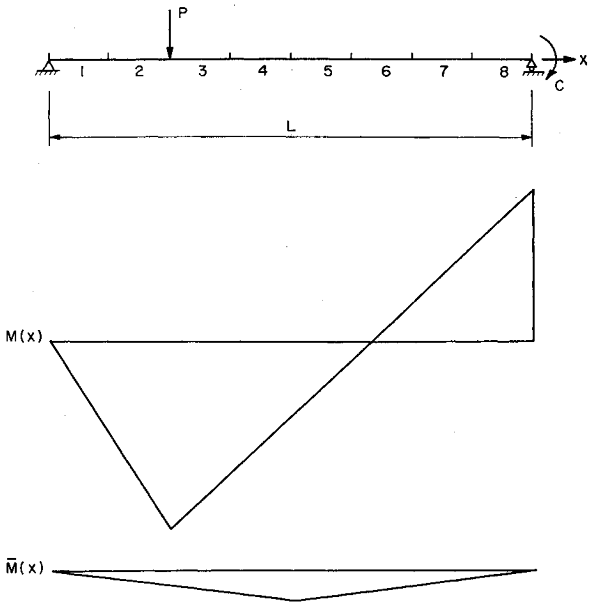

Fig. 1 Beam studied and bending moments M and $\bar{M}$.

to introduce an upper bound on the bending stress as an additional constraint. The aim of the present Note is to show that these behavioral constraints are complementary, in the sense that the design of each element of a beam of piecewise constant cross section is governed either by the displacement constraint or by the stress constraint. The derivation of the optimality conditions essentially follows that given by Chern and Prager[3] for a statically determinate truss.

Optimality Conditions

Consider a sandwich beam of length L, divided into n elements of constant bending stiffness s_i $(i = 1,2,\ldots,n)$. Let $x_0 = 0,x_1,x_2,\ldots,x_n = L$ denote the abscissae of the points of division and $l_i = x_{i-1} - x_i$, the length of the ith element. Furthermore, let $M(x)$ and $\bar{M}(x)$ $(0 \leq x \leq L)$ be the bending moments under the given loads and a unit load (couple) applied at the point x_j whose displacement (rotation) is prescribed, respectively. We introduce the following notations:

$$m_i^2 = \frac{1}{l_i} \int_{x_{i-1}}^{x_i} M^2(x)dx$$
$$\bar{m}_i = \frac{1}{l_i} \int_{x_{i-1}}^{x_i} M(x)\bar{M}(x)dx \tag{1}$$

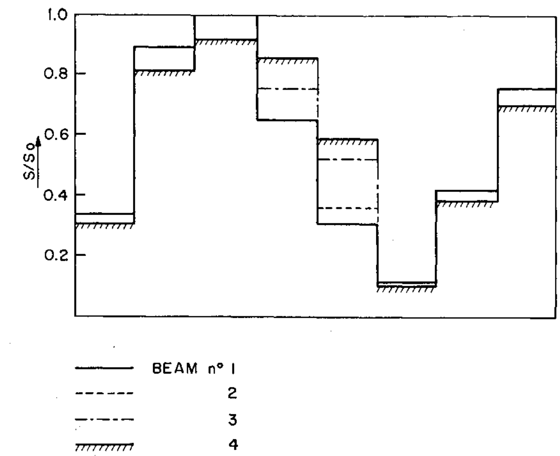

Fig. 2 Optimal stiffness.

Since the beam is assumed to be of sandwich-type, the stresses in the cover sheets of the ith element are given by $M_i(x)H/s_i$ $(x_{i-1} \leq x \leq x_i)$, $2H$ being the constant height of the beam, and the mean square stress σ_i^2 of this element is defined as

$$\sigma_i^2 = \frac{H^2}{l_i} \int_{x_{i-1}}^{x_i} \frac{M_i^2(x)}{s_i^2}\, dx = \frac{H^2 m_i^2}{s_i^2} \tag{2}$$

It follows from the definition of $\bar{M}(x)$ and from the virtual work equation that the displacement (rotation) of the point x_j is

$$\delta = \sum_i \int_{x_{i-1}}^{x_i} M(x)\, \frac{\bar{M}(x)}{Es_i}\, dx = \sum_i \frac{l_i \bar{m}_i}{Es_i} \tag{3}$$

where E denotes Young's modulus.

If $H\hat{\sigma}_i$ denotes the upper bound of the mean square stress of the ith element and $\hat{\delta}$ the upper bound of the displacement (rotation) at the point x_j, the behavioral constraints may be written as

$$\sum_i \frac{l_i \bar{m}_i}{Es_i} - \hat{\delta} \leq 0$$
$$m_i^2/s_i^2 - \hat{\sigma}_i^2 \leq 0, \; (i = 1,2,\ldots,n) \tag{4}$$

Since the specific structural weight of the beam is proportional to its bending stiffness,

$$V = \sum_i l_i s_i \tag{5}$$

is a measure of the total structural weight. The problem that has to be solved may thus be stated as follows: Find s_i $(i = 1,2,\ldots,n)$ to minimize Eq. (5) subject to the constraints (4).

The Lagrangian function to be minimized is therefore

$$\phi(s_i,\lambda_0,\lambda_i) = \sum_i l_i s_i + \lambda_0 \left(\sum_i \frac{l_i \bar{m}_i}{Es_i} - \hat{\delta} \right) + \sum_i \lambda_i \left(\frac{m_i^2}{s_i^2} - \hat{\sigma}_i^2 \right) \tag{6}$$

where λ_0 and λ_i are non-negative multipliers.

Since $\bar{m}_i$ may be negative, this function is not convex but if we introduce the new variables $r_i = 1/s_i$, then the corresponding Lagrangian function

$$\Psi(r_i,\lambda_0,\lambda_i) = \sum_i \frac{l_i}{r_i} + \lambda_0 \left(\sum_i \frac{l_i \bar{m}_i r_i}{E} - \hat{\delta} \right) + \sum_i \lambda_i (m_i^2 r_i^2 - \hat{\sigma}_i^2) \tag{7}$$

is convex and it follows from the Kuhn-Tucker theorem[4] that the values of r_i,λ_0,λ_i which minimize Ψ under the conditions $r_i > 0, \lambda_0 \geq 0, \lambda_i \geq 0$ must fulfill the conditions

$$\partial\Psi/\partial r_i = -l_i/r_i^2 + \lambda_0 l_i \bar{m}_i/E + 2\lambda_i m_i^2 r_i = 0 \tag{8a}$$

$$\partial\Psi/\partial\lambda_0 = \sum_i l_i \bar{m}_i r_i/E - \hat{\delta} \leq 0$$

according to whether $\lambda_0 \gtreqless 0$ (8b)

Table 1 Numerical data for the beams studied

Beam $n°$	$H\hat{\sigma}$ t/m^2	$\hat{\delta}$ m	$H\sigma$ max t/m^2	δ m	s_0 m^4
1	4 10^5	1 10^{-2}	4 10^5	0.624 10^{-2}	2.67 10^{-3}
2	5	1	5	0.780	2.14
3	6	1	6	0.936	1.78
4	7	1	6.41	1	1.67

Table 2 Division in groups and optimal stiffnesses

Beam $n°$		Iter.	Elem. 1	2	3	4	5	6	7	8	Volume V/V_0
1	Groups	1	2	2	2	2	2	1	1	1	
		2	1	1	1	2	2	1	1	1	
		3	1	1	1	1	2	1	1	1	
		4	1	1	1	1	1	1	1	1	
	s/s_0		0.338	0.894	1.000	0.652	0.310	0.117	0.422	0.767	0.5624
2	Groups	1	2	2	2	2	2	1	1	1	
		2	1	1	1	2	2	1	1	1	
		3	1	1	1	1	2	1	1	1	
	s/s_0		0.338	0.894	1.000	0.652	0.367	0.117	0.422	0.767	0.5697
3	Groups	1	2	2	2	2	2	1	1	1	
		2	1	1	1	2	2	1	1	1	
	s/s_0		0.338	0.894	1.000	0.759	0.522	0.117	0.422	0.767	0.6024
4	Groups	1	2	2	2	2	2	1	1	1	
		2	1	1	2	2	2	1	1	1	
		3	1	1	1	2	2	1	1	1	
	s/s_0		0.309	0.818	0.915	0.860	0.592	0.107	0.386	0.703	0.5864

$$\partial\Psi/\partial\lambda_i = m_i^2 r_i^2 - \hat{\sigma}_i^2 \leqq 0 \quad \text{according to whether } \lambda_i \gtreqqless 0 \tag{8c}$$

To discuss these conditions, we have to distinguish the two following cases:

1) Displacement constraint is not relevant: $\lambda_0 = 0$. It follows from Eq. (8a) that, in this case,

$$\lambda_i = l_i/2m_i^2 r_i^3 > 0 \text{ for all } i$$

and accordingly, Eq. (8c) yields the condition

$$m_i^2/s_i^2 = \hat{\sigma}_i^2 \tag{9}$$

which is the optimality condition for a fully stressed sandwich beam.

2) Displacement constraint is relevant: $\lambda_0 > 0$. In that case, we divide the elements into the groups G_1 and G_2, according to whether the stress constraint is or is not relevant. Thus,

$$\lambda_i > 0 \text{ for } i\epsilon G_1,\ \lambda_i = 0 \text{ for } i\epsilon G_2$$

For the elements of group 2, we have according to Eq. (8a)

$$\bar{m}_i/s_i^2 = E/\lambda_0 = \sigma_0^2 \text{ (say)} \tag{10}$$

which is the optimality condition given by Shield and Prager for prescribed displacement[5]; moreover, Eq. (8c) yields $m_i^2/s_i^2 < \hat{\sigma}_i^2$. For the elements of group 1, it follows from Eq. (8c) that

$$m_i^2/s_i^2 = \hat{\sigma}_i^2 \tag{11}$$

and substitution of this condition into Eq. (8a) gives

$$\bar{m}_i/s_i^2 = \sigma_0^2(1 - 2\lambda_i\hat{\sigma}_i^2/l_i s_i) < \sigma_0^2$$

To summarize, the optimality conditions are

$$\begin{aligned} &m_i^2/s_i^2 = \hat{\sigma}_i^2 \text{ and } \bar{m}_i/s_i^2 < \sigma_0^2, \text{ for } i\epsilon G_1 \\ &m_i^2/s_i^2 < \hat{\sigma}_i^2 \text{ and } \bar{m}_i/s_i^2 = \sigma_0^2, \text{ for } i\epsilon G_2 \end{aligned} \tag{12}$$

Note that all elements for which $\bar{m}_i \leq 0$ must belong to G_1, because they can obviously not fulfill the condition $\bar{m}_i/s_i^2 = \sigma_0^2$; but, of course, this does not mean that all elements for which $\bar{m}_i > 0$ belong to G_2.

Suppose we know the division of elements into G_1 and G_2. The stiffnesses are then given by

$$s_i = m_i/\hat{\sigma}_i, \text{ for } i\epsilon G_1 \tag{13a}$$

on the other hand, the displacement constraint (3) and the optimality conditions yield

$$s_i = \bar{m}_i^{1/2}\left[\sum_{j\epsilon G_2} l_j \bar{m}_j^{1/2}\right]\Big/\left[\hat{\delta}E - \sum_{j\epsilon G_1} m_j/s_j\right], \text{ for } i\epsilon G_2 \tag{13b}$$

Division of Elements into G_1 and G_2

Since the beam is supposed to be statically determinate, m_i and $\bar{m}_i$ do not depend on the stiffness and if we know the division of elements into G_1 and G_2, then relations (13) give the optimal design.

To find this division, we use the following iterative method: We choose as a first guess for G_1, all elements for which $\bar{m}_i \leq 0$ and we compute s_i according to Eq. (13). This first approximation fulfills the displacement constraint but may violate the stress constraint for some elements of G_2. For this reason, we have to calculate the mean square stress $\sigma_i = m_i/s_i$ for $i\epsilon G_2$ and to put into G_1 all elements of G_2 for which $\sigma_i^2 > \hat{\sigma}_i^2$. This gives us a new tentative division into groups for which we use Eq. (13) to find a new design, check stresses of elements of G_2, and so on.

Note that an element which belongs to G_1 at one step, cannot fall into G_2 later. This means that the number of trial divisions is less or equal than the number of elements of G_2 of the first guess. Practically, it appears that even if the first tentative division is far from being correct, very few iterations seem to be required, as shown in the following example.

Example

Consider the beam in Fig. 1, which carries a concentrated load P and a couple C. The displacement at the center of the span is prescribed together with an upper bound on the mean square stress. The computation was carried out with the following numerical values:

$$L = 8m, \quad P = 10t, \quad C = 10tm, \quad E = 210^7 t/m^2, \quad \hat{\delta} = 10^{-2}m$$

and for the various values of $H\hat{\sigma}$:

$$H\hat{\sigma} = 4\ 10^4,\ \ 5\ 10^4,\ 6\ 10^4,\ 7\ 10^4 t/m^2$$

Table 1 gives for each value of $\hat{\sigma}$ the characteristics of the uniform beam that fulfills the behavioral constraints. For each value of $\hat{\sigma}$, Table 2 gives the various tentative division of elements into G_1 and G_2 and the optimal stiffness scaled by that of the corresponding uniform beam. It appears in Table 2 that if the displacement constraint is not relevant, all elements will belong to G_1 after a few iterations and consequently, the beam will be fully stressed, so that both cases distinguished in Sec. 2 may be computed by the same technique. Figure 2 gives the optimal design for various values of $\hat{\sigma}$.

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A Specialization of Jones' Generalization of the Direct-Stiffness Method of Structural Analysis

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Jones[1] has presented a variational formulation which accounts for displacement discontinuities across element boundaries. The purpose of this Note is to present a specialized version of his results. For classical elasticity, Jones considered the functional

$$\pi(u_i,\alpha^i,\beta^i) = \int_V \left(\tfrac{1}{2}C^{ijrs}u_{i,j}u_{r,s} - f^iu_i\right)dv - \int_{S_{E\bar{T}}} \bar{T}^iu_ids - \int_{S_{E\bar{U}}} \alpha^i(u_i - \bar{u}_i)ds - \int_{S_I} \beta^i(u_i{}^1 - u_i{}^2)ds - \int_{S_I} \bar{T}^i\tfrac{1}{2}(u_i{}^1 + u_i{}^2)ds$$

where u_i = the covariant components of the displacement vector; $\alpha^i\beta^i$ = Lagrange multipliers; C^{ijrs} = the contravariant components of the elasticity tensor; f^i = the contravariant components of the body force vector; T^i = the contravariant components of the surface traction vector; V = the volume of the body; S_E = the exterior surface of the body; S_I = an interior surface in the body; $S_{E\bar{T}}$ = that portion of the exterior surface where a traction $\bar{T}^i$ is prescribed; $S_{E\bar{U}}$ = that portion of the exterior surface where a displacement u_i is prescribed.

The superscripts and subscripts 1 and 2 in the interior surface integrals denote quantities on either side of the surface. The commas denote covariant derivatives. Figure 1 depicts a body with an interior surface.

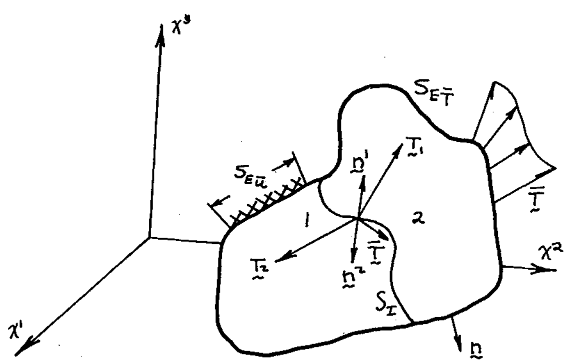

Fig. 1 Body with an interior surface.

Taking the variation of π results in

$$\delta\pi = \int_V (C^{ijrs}u_{r,s}\delta u_{i,j} - f^i\delta u_i)dv - \int_{S_{E\bar{T}}} \bar{T}^i\delta u_ids - \int_{S_{E\bar{U}}} \alpha^i\delta u_ids - \int_{S_{E\bar{U}}} (u_i - \bar{u}_i)\delta\alpha^ids - \int_{S_I} \beta^i\delta(u_i{}^1 - u_i{}^2)ds - \int_{S_I} (u_i{}^1 - u_i{}^2)\delta\beta^ids - \int_{S_I} \bar{T}^i\tfrac{1}{2}\delta(u_i{}^1 + u_i{}^2)ds$$

Applying the divergence theorem results in

$$\delta\pi = -\int_V [(C^{ijrs}u_{r,s})_{,j} + f^i]\delta u_idv + \int_{S_{E\bar{T}}}(T^i - \bar{T}^i)\delta u_ids + \int_{S_{E\bar{U}}} (T^i - \alpha^i)\delta u_ids - \int_{S_{E\bar{U}}} (u_i - \bar{u}_i)\delta\alpha^ids + \int_{S_I} \tfrac{1}{2}(T_1{}^i + T_2{}^i - \bar{T}^i)\delta(u_i{}^1 + u_i{}^2)ds - \int_{S_I} (u_i{}^1 - u_i{}^2) \times \delta\beta^ids - \int_{S_I} [\beta_i - (T_1{}^i - T_2{}^i)/2]\,\delta(u_i{}^1 - u_i{}^2)ds$$

where $T^i = C^{ijrs}u_{r,s}n_j$. This result can be specialized in the following way. The α^i and β^i are selected as $\alpha^i = T^i = C^{ijrs}u_{r,s}n_j$ on $S_{E\bar{U}}$, $\beta^i = \frac{1}{2}(T_1{}^i - T_2{}^i) = \frac{1}{2}(C^{ijrs}u_{r,s}{}^1n_j{}^1 - C^{ijrs}u_{r,s}{}^2n_j{}^2)$ on S_I. Making these substitutions in the functional π results in a new functional Γ:

$$\Gamma(u_i) = \int_V \left(\tfrac{1}{2}C^{ijrs}u_{i,j}u_{r,s} - f^iu_i\right)dv - \int_{S_{E\bar{T}}} \bar{T}^iu_ids - \int_{S_{E\bar{U}}} T^i(u_i - \bar{u}_i)ds - \int_{S_I} \tfrac{1}{2}(T_1{}^i - T_2{}^i)(u_i{}^1 - u_i{}^2)ds - \int_{S_I} \bar{T}^i\tfrac{1}{2}(u_i{}^1 + u_i{}^2)ds$$

Taking the variation results in

$$\delta\Gamma = \int_V (C^{ijrs}u_{r,j}\delta u_{i,s} - f^i\delta u_i)dv - \int_{S_{E\bar{T}}} \bar{T}^i\delta u_ids - \int_{S_{E\bar{U}}} (u_i - \bar{u}_i)\delta T^ids - \int_{S_{E\bar{U}}} T^i\delta u_ids - \int_{S_I} \tfrac{1}{2}(u_i{}^1 - u_i{}^2)\delta(T_1{}^i - T_2{}^i)ds - \int_{S_I} \tfrac{1}{2}(T_1{}^i - T_2{}^i) \times \delta(u_i{}^1 - u_i{}^2)ds - \int_{S_I} \bar{T}^i\tfrac{1}{2}\delta(u_i{}^1 + u_i{}^2)ds$$

Using the divergence theorem results in

$$\delta\Gamma = -\int_V [(C^{ijrs}u_{r,s})_{,j} + f^i]\delta u_idv + \int_{S_{E\bar{T}}} (C^{ijrs}u_{r,s}n_j - \bar{T}^i)\delta u_ids - \int_{S_{E\bar{U}}} (u_i - \bar{u}_i)\delta T^ids - \int_{S_I} \tfrac{1}{2}(u_i{}^1 - u_i{}^2) \times \delta(T_1{}^i - T_2{}^i)ds + \int_{S_I} \tfrac{1}{2}(T_1{}^i + T_2{}^i - \bar{T}^i)\delta(u_i{}^1 + u_i{}^2)ds$$

Thus, if an arbitrary choice of displacements is made

$$u_i = \sum_{\alpha=1}^{n} a_{i\alpha}g_\alpha(x^i)$$

then extremizing Γ for this choice results in the approximate satisfaction of equilibrium in the interior, $(C^{ijrs}u_{r,s})_{,j} + f^i = 0$, stress jump, surface load equilibrium on the interior surfaces, $T_1{}^i + T_2{}^i - \bar{T}^i = 0$, the static or stress boundary conditions, $T^i = \bar{T}^i$, the displacement boundary conditions, $u^i = \bar{u}^i$, and the displacement discontinuities on the interior surfaces, $u_i{}^1 - u_i{}^2 = 0$. Thus, in the Direct-Stiffness method, element displacements selected for use with this functional need not be "compatible" along element boundaries. The new functional Γ does not require an independent assumption on the behavior of the Lagrange multipliers α_i and β_i found in the functional π. In essence, the field equations of the functional π involving the Lagrange multipliers have been satisfied exactly. The functionals π and Γ correspond to the functionals M and N, respectively, discussed by Prager.[2]

Received January 11, 1971. This work was supported by the U.S. Atomic Energy Commission.

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